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FORECASTING DISCRETE STOCK AND FLOW DATA GENERATED BY A SECOND ORDER CONTINUOUS TIME SYSTEM

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Abstract—This paper extends recent results on forecasting discrete stock and flow data generated by a continuous time model formulated as a system of second order stochastic differential equations. An algorithm is presented for the computation of the forecasts, and an expression for the asymptotic mean square error matrix of the forecasts is also derived. These results, when combined with the estimation algorithm in [1], provide a computationally efficient method of estimation, forecasting and inference in continuous time systems.

1. INTRODUCTION

In a recent paper [2], an algorithm was presented for computing forecasts of discrete stock and flow data generated by a system of second order stochastic differential equations with constant coefficients. Such models are extremely important in macroeconomic modelling because although aggregate data is observed only at discrete intervals of time, typically on a quarterly basis, such data represents the actions of millions of individuals who make their decisions at different points in time. The effect of these individual actions makes aggregate data liable to almost continual change, which can only realistically be captured with a continuous time model formulated as a system of stochastic differential equations.

The algorithm in [2] produces forecasts which are exact maximum likelihood estimates of the expectations of the post-sample observations, conditional on the information in the sample, under the assumption that the innovations are Gaussian. In order to produce these forecasts, it is necessary to estimate the parameters of the continuous time system from a sample of discrete observations. The observations themselves fall into two distinct categories. The first type relates to stock variables, such as the money supply and inventory holdings, which are measured at points in time at a particular frequency, for example quarterly. The second type of observations relates to flow variables, such as consumption and income, which are measured as integrals over units of time (quarters) of the underlying rates of flow. It is important, for purposes of estimation, to have an algorithm which takes this distinction between the different types of variables into account and one which is able to relate the parameters of the continuous time model to the discrete observations. This has been achieved using two different approaches.

The first approach detailed in [1] derives a discrete vector autoregressive moving average model with exogenous variables (VARMAX model) which holds exactly if the data are generated by the continuous time model. The algorithm preposed in [1] is computationally efficient and exploits the fact that many of the relevant matrices are sparse in nature. An alternative approach given in [3,4] expresses the continuous time model in state space form and makes extensive use of the Kalman filter. However, the approach in [1] has been shown in [5] to be even more efficient computationally than the Kalman filtering techniques.

The purpose of this paper is to make a further contribution to the issue of forecasting discrete stock and flow data generated by an open second order stochastic differential equation system.

In particular, suitable expressions are derived for computing the asymptotic mean square error (AMSE) matrix of the dynamic multi-step ahead predictions which can be used in conjunction with the efficient estimation algorithm in [1]. The AMSE matrix is important for making inferences regarding the forecasts produced by the estimated model, for example in the construction of confidence intervals. In deriving these results, different (but equivalent) expressions are given for the forecasts to those used in [2]. These methods have been used recently by the author in applied studies [6,7] with considerable success.

The paper is organised as follows. The continuous time model is presented in Section 2 and the estimation method is outlined with references to the appropriate sources. The expressions for the forecasts are given in Section 3, and the derivation of the AMSE matrix proceeds in Section 4. Some concluding comments are given in Section 5.

2. THE MODEL

Consider the following system of second order stochastic differential equations:

$$\begin{aligned} d[Dx(t)] &= [A_1(\theta)Dx(t) + A_2(\theta)x(t) + B_2(\theta)D^2z(t) \\ &\quad + B_1(\theta)Dz(t) + B_0(\theta)z(t)]dt + \zeta(dt) \quad (t \leq 0) \\ x(0) &= y_1, \quad Dx(0) = y_2 \end{aligned} \tag{1}$$

where D denotes the time derivative $\frac{d}{dt}$, $x(t)$ is an $n \times 1$ vector of endogenous variables, $z(t)$ is an $m \times 1$ vector of exogenous variables, and the matrices A_1, A_2, B_0, B_1 and B_2 are known functions of an underlying $p \times 1$ vector θ of structural parameters ($p \leq 2n^2 + 3\,\mathrm{nm}$). We will assume that the series $\{x(t)\}$ and $\{z(t)\}$ are stationary, and that the matrix A_2 is nonsingular. Estimation and inference in the case where zero roots occur has recently been considered in [8] and results in non-standard asymptotic theory. We will also regard the initial conditions y_1 and y_2 as being fixed (non-random) vectors. With regard to the vector $\zeta(dt)$ of innovations in (1) we make the following assumption.

ASSUMPTION 1. *$\zeta = [\zeta_1, \dots, \zeta_n]'$ is a vector of random measures defined on all subsets of the half line $0 < t < \infty$ with finite Lebesgue measure, such that $E[\zeta(dt)] = 0, E[\zeta(dt)\zeta(dt)'] = dt\,\Sigma(\mu)$, where Σ is a positive definite matrix whose elements are known functions of a $q \times 1$ vector μ of unknown parameters ($q \leq n(n+1)/2$), and $E[\zeta_i(\Delta_1)\zeta_j(\Delta_2)] = 0 (i,j = 1, \dots, n)$ for any disjoint sets Δ_1 and Δ_2 on the half line $0 < t < \infty$ (see [9, p. 1157] for a discussion of random measures and their application to continuous time models).*

Equation (1) is interpreted as meaning that

$$\begin{aligned} Dx(t) - Dx(0) = \int_0^t & [A_1(\theta)Dx(r) + A_2(\theta)x(r) + B_2(\theta)D^2z(r) \\ & + B_1 Dz(r) + B_0 z(r)]\, dr + \int_0^t \zeta(dr), \end{aligned}$$

where

$$\int_0^t \zeta(dr) = \zeta\,[0,t].$$

The interpretation of random integrals of the type given above is dealt with in [10].

In order to estimate the parameters of interest from a sample of T discrete observations, it is necessary to derive from (1) a discrete model which holds exactly if (1) is the true process generating the data. This is complicated by the fact that the variables are generally mixtures of stocks and flows. Denoting a stock variable by a superscript s and a flow variable by a superscript f, the observed discrete data series will be arranged (without loss of generality) into the series

$\{x_t\}$ and $\{z_t\}$, given by

$$x_t' = \left[\{x^s(t) - x^s(t-1)\}' : \left\{\int_{t-1}^{t} x^f(r)\,dr\right\}'\right],$$

$$z_t' = \left[\frac{1}{2}\{z^s(t) + z^s(t-1)\}' : \left\{\int_{t-1}^{t} z^f(r)\,dr\right\}'\right],$$

where x_t is of dimension $(n^s + n^f) \times 1$, z_t is of dimension $(m^s + m^f) \times 1$, $n^s + n^f = n$ and $m^s + m^f = m$. It can be shown that the discrete observations generated by (1) satisfy the vector autoregressive moving average model given by (2) to (7) below:

$$x_1 = G_{11}y_1 + G_{12}y_2 + E_{11}z_1 + E_{12}z_2 + E_{13}z_3 + \eta_1, \tag{2}$$
$$x_2 = C_{11}x_1 + G_{21}y_1 + G_{22}y_2 + E_{21}z_1 + E_{22}z_2 + E_{23}z_3 + \eta_2, \tag{3}$$
$$x_t = F_1x_{t-1} + F_2x_{t-2} + E_0z_t + E_1z_{t-1} + E_2z_{t-2} + \eta_t, \qquad t = 3,\dots,T, \tag{4}$$

where

$$\eta_1 = \int_0^1 K_{11}(1-r)\zeta(dr), \tag{5}$$
$$\eta_2 = \int_0^1 K_{21}(1-r)\zeta(dr) + \int_1^2 K_{22}(2-r)\zeta(dr), \tag{6}$$
$$\eta_t = \int_{t-1}^{t} K_1(t-r)\zeta(dr) + \int_{t-2}^{t-1} K_2(t-1-r)\zeta(dr) + \int_{t-3}^{t-2} K_3(t-2-r)\zeta(dr), \qquad t = 3,\dots,T, \tag{7}$$

where the matrices G_{11}, G_{12}, G_{21}, G_{22}, C_{11}, F_1 and F_2, and functions $K_{11}(r)$, $K_{21}(r)$, $K_{22}(r)$, $K_1(r)$, $K_2(r)$ and $K_3(r)$ are given in Theorems 2.1 and 2.2 of [1] and the matrices E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}, E_0, E_1, and E_2 are given in Theorems 1 and 2 of [11]. In fact, the matrices and functions entering into Equations (2) to (7) are complicated transcendental functions of the underlying parameters of the continuous time model.

From Equations (5) to (7), we can express the covariance properties of the model more concisely as follows:

$$\begin{array}{lll} E(\eta_1\eta_1') = \Omega_{11}, & E(\eta_2\eta_1') = \Omega_{21}, & E(\eta_3\eta_1') = \Omega_{31}, \\ E(\eta_2\eta_2') = \Omega_{22}, & E(\eta_3\eta_2') = \Omega_{32}, & E(\eta_4\eta_2') = \Omega_{42}, \\ E(\eta_t\eta_t') = \Omega_0, & E(\eta_t\eta_{t-1}') = \Omega_1, & E(\eta_t\eta_{t-2}') = \Omega_2, \\ t = 3,\dots,T. & t = 4,\dots,T. & t = 5,\dots,T. \end{array} \tag{8}$$

Each of the $n \times n$ matrices above depends on both θ and μ.

Given the discrete data series $\{x_t\}_1^T$ and $\{z_t\}_1^T$, estimates of θ and μ can be obtained by maximising the likelihood function derived under the assumption that $\{\eta_t\}_1^T$ is distributed as multivariate normal. In addition to θ and μ, the initial state vector $y' \equiv \left[y_1^{f\prime} y_2'\right]$ can also be regarded as a supplementary vector to be estimated (note that y_1^s forms part of the data and is therefore known). Equations (2) to (4) can then be more concisely expressed as the matrix system

$$F(\theta)x = E(\theta, y)z + \eta(\theta, \mu) \tag{9}$$

where $x = [x_1', \dots, x_T']'$ is of dimension $nT \times 1$, $z = \left[\text{vec}\,(I_n)', y_1^{s\prime}, z_1', \dots, z_T'\right]'$ is of dimension $(n^2 + n^s + mT) \times 1$, $\eta = [\eta_1', \dots, \eta_T']'$ is of dimension $nT \times 1$, and where the following matrices are defined:

$$F(\theta) = \begin{bmatrix} I & 0 & 0 & \dots & \dots & \dots & 0 \\ -C_{11} & I & 0 & & & & \vdots \\ -F_2 & -F_1 & I & 0 & & & \vdots \\ 0 & -F_2 & -F_1 & I & & & \vdots \\ \vdots & & & & & & \vdots \\ 0 & \dots & \dots & 0 & -F_2 & -F_1 & I \end{bmatrix} \quad (nT \times nT),$$

$$E(\theta, y) = \begin{bmatrix} I_n \otimes Y_1 & G_{11}^S & E_{11} & E_{12} & E_{13} & 0 & \dots & \dots & \dots & 0 \\ I_n \otimes Y_2 & G_{21}^S & E_{21} & E_{22} & E_{23} & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & E_2 & E_1 & E_0 & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & E_2 & E_1 & E_0 & 0 & \dots & \dots & 0 \\ \vdots & & & & & & \vdots & & & \vdots \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & E_2 & E_1 & E_0 \end{bmatrix} \quad (nT \times n^2 + n^S + mT),$$

and where

$$\left[G_{11}^S : G_{11}^f\right] = G_{11}, \qquad \left[G_{21}^S : G_{21}^f\right] = G_{21},$$
$$Y_1' = G_{11}^f\, y_1^f + G_{12}\, y_2, \quad Y_2' = G_{21}^f\, y_1^f + G_{22}\, y_2,$$

I_n is an $n \times n$ identity matrix, and vec(.) is the operator which stacks the columns of a $j \times k$ matrix vertically to form a $jk \times 1$ column vector.

The system of equations represented by (9) is, in fact, in the form of a VARMAX model. Noting that $E(\eta) = 0$ and $E(\eta\eta') = \Omega$, where Ω is a block Toeplitz matrix containing the expressions in Equation (8), we may express η as $\eta = M\varepsilon$, where M is the real lower triangular diagonal matrix satisfying $MM' = \Omega$, with no more than $3n$ non-zero elements in any row, and ε is a vector whose elements are standard normal variates. Equation (9) can thus be expressed as the system

$$F(\theta)x = E(\theta, y)z + M(\theta, \mu)\varepsilon. \tag{10}$$

It is interesting to note that the moving average representation for $t = 3, \dots, T$ is of the form

$$\eta_t = M_{t,t}\, \varepsilon_t + M_{t,t-1}\, \varepsilon_{t-1} + M_{t,t-2}\, \varepsilon_{t-2},$$

where the coefficient matrices depend on time. But it can be shown that the matrices $M_{t,t}$, $M_{t,t-1}$ and $M_{t,t-2}$ converge rapidly to constant matrices, denoted M_0, M_1 and M_2 respectively; see [12] for details.

Although there is no way of obtaining consistent estimates of the vector y of initial conditions, it can be shown that $T^{1/2}(\hat{\alpha} - \alpha)$, where $\hat{\alpha}$ is the Gaussian estimator of $\alpha = [\theta'\mu']'$, will have a limiting normal distribution with mean vector 0 and covariance matrix denoted by Q (see [12] for details). As the results derived in the next section are asymptotic, it would be justifiable to ignore the influence of the initial state vector y and obtain asymptotically equivalent results. But the finite sample performance of the forecasts may be affected by this omission, especially in view of the sample sizes typically available to econometricians, so the influence of y is included for completeness.

3. COMPUTATION OF THE FORECASTS

We will assume that α and y have been estimated from a sample of T discrete observations by the method of maximum likelihood and that it is desired to produce forecasts of the vector x_t for periods $T+1$ through to $T+h$. We begin by extending the system of equations given by (10) to cover the h forecast periods as follows:

$$\begin{bmatrix} F & 0 \\ F^* & F_h \end{bmatrix} \begin{bmatrix} x \\ x_h \end{bmatrix} = \begin{bmatrix} E & 0 \\ E^* & E_h \end{bmatrix} \begin{bmatrix} z \\ z_h \end{bmatrix} + \begin{bmatrix} M & 0 \\ M^* & M_h \end{bmatrix} \begin{bmatrix} \varepsilon \\ \varepsilon_h \end{bmatrix} \tag{11}$$

where $x_h = [x'_{T+1}, \ldots, x'_{T+h}]'$ is of dimension $nh \times 1$, $z_h = [z'_{T+1}, \ldots, z'_{T+h}]'$ is of dimension $mh \times 1$, and $\varepsilon_h = [\varepsilon'_{T+1}, \ldots, \varepsilon'_{T+h}]'$ is of dimension $nh \times 1$. It is assumed that z_h is observable, which is typically the case when forecasting performance is assessed for an estimated model by retaining some observations at the end of the sample for this purpose, as in [6,7]. The additional matrices in (11) are defined as follows:

$$F^*(\theta) = \begin{bmatrix} 0 & \ldots & 0 & -F_2 & -F_1 \\ 0 & \ldots & 0 & 0 & -F_2 \\ 0 & & & & 0 \\ \vdots & & & & \vdots \\ 0 & \ldots & \ldots & \ldots & 0 \end{bmatrix}, \quad F_h(\theta) = \begin{bmatrix} I & 0 & 0 & \ldots & \ldots & \ldots & 0 \\ -F_1 & I & 0 & \ldots & \ldots & \ldots & 0 \\ -F_2 & -F_1 & I & & & & \vdots \\ \vdots & \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \ldots & -F_2 & -F_1 & I \end{bmatrix},$$

$$E^*(\theta) = \begin{bmatrix} 0 & \ldots & 0 & E_2 & E_1 \\ 0 & \ldots & 0 & 0 & E_2 \\ 0 & & & & 0 \\ \vdots & & & & \vdots \\ 0 & \ldots & \ldots & \ldots & 0 \end{bmatrix}, \quad E_h(\theta) = \begin{bmatrix} E_0 & 0 & 0 & \ldots & \ldots & \ldots & 0 \\ E_1 & E_0 & 0 & \ldots & \ldots & \ldots & 0 \\ E_2 & E_1 & E_0 & & & & \vdots \\ \vdots & \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \ldots & E_2 & E_1 & E_0 \end{bmatrix},$$

$$M^*(\theta,\mu) = \begin{bmatrix} 0 & \ldots & 0 & M_2 & M_1 \\ 0 & \ldots & 0 & 0 & M_2 \\ 0 & & & & 0 \\ \vdots & & & & \vdots \\ 0 & \ldots & \ldots & \ldots & 0 \end{bmatrix}, M_h(\theta,\mu) = \begin{bmatrix} M_0 & 0 & 0 & \ldots & \ldots & \ldots & 0 \\ M_1 & M_0 & 0 & \ldots & \ldots & \ldots & 0 \\ M_2 & M_1 & M_0 & & & & \vdots \\ \vdots & \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \ldots & M_2 & M_1 & M_0 \end{bmatrix},$$

where $F^*(\theta)$ and $M^*(\theta,\mu)$ are of dimension $nh \times nT$, $F_h(\theta)$ and $M_h(\theta,\mu)$ are of dimension $nh \times nh$, $E^*(\theta)$ is of dimension $nh \times mT$ and $E_h(\theta)$ is of dimension $nh \times mh$. Note that none of these additional matrices depends on y. Taking the last nh equations in (11), we obtain

$$F^* x + F_h\, x_h = E^* z + E_h\, z_h + M^* \varepsilon + M_h\, \varepsilon_h, \tag{12}$$

where the dependence of the matrices on θ and μ has been suppressed for notational convenience. Solving for x_h yields the expression

$$x_h = F_h^{-1}\left[-F^* x + E^* z + E_h\, z_h + M^* \varepsilon + M_h\, \varepsilon_h\right], \tag{13}$$

which highlights the fact that in models of this type, the value of $x_{T+j} (1 \leq j \leq h)$ depends on the entire past history of the variables.

It is convenient for subsequent work to define two estimators of x_h, namely the optimal estimator, which uses knowledge of the true parameter values, and the practical estimator, using the estimated parameter values. Thus, the optimal h-step ahead predictor based on a sample of size T is given by

$$x_{T,h} = F_h^{-1}\left[-F^* x + E^* z + E_h\, z_h + M^* \varepsilon\right] \tag{14}$$

and the practical estimator of x_h, which relies on $\hat{\alpha}$ and the sample residuals $\hat{\varepsilon}$, is given by

$$\hat{x}_{T,h} = \hat{F}_h^{-1}\left[-\hat{F}^* x + \hat{E}^* z + \hat{E}_h\, z_h + \hat{M}^* \hat{\varepsilon}\right], \tag{15}$$

where a '$\hat{}$' denotes that the matrix is based on the estimate $\hat{\alpha} = \left[\hat{\theta}' \hat{\mu}'\right]'$ (so that, for example, $\hat{F}_h = F_h(\hat{\theta})$etc.), and where $\hat{\varepsilon}$ satisfies the system

$$\hat{F} x = \hat{E} z + \hat{M} \hat{\varepsilon}. \tag{16}$$

The forecasts given by (15) may be computed using the following algorithm:

(i) Compute F^*, E^*, E_h, M^* and F_h using the matrices F_1, F_2, E_0, E_1, E_2, M_1 and M_2 which will have been computed in the estimation of $\hat{\alpha}$ and $\hat{y}$.
(ii) Compute F_h^{-1}.
(iii) Compute $\hat{x}_{T,h}$ using Equation (15).

Note that the inversion of F_h in step (ii) is relatively straightforward since $\det(F_h) = 1$ and that F_h^{-1} is lower triangular. The computations are based on just two matrices, F_1 and F_2, which will already have been computed as part of the estimation procedure. Indeed, the submatrices entering into F^*, E^*, E_h and M^* will also have been previously calculated, so that the forecasting algorithm is easily incorporated with the estimation algorithm in [1]. The practical forecasts obtained by use of (15) may also be calculated using a simple recursion as in [2], but the equivalent matrix representation used here is more convenient for obtaining the AMSE matrix in the next section.

4. THE ASYMPTOTIC MEAN SQUARE ERROR MATRIX

The derivation of the AMSE matrix of forecasts is simplified by dividing the forecast error into two components as follows:

$$x_h - \hat{x}_{T,h} = (x_h - x_{T,h}) - (\hat{x}_{T,h} - x_{T,h}), \tag{17}$$

which makes use of the optimal predictor $x_{T,h}$. Since the two components on the right hand side of (17) are mutually independent, the asymptotic mean square error matrix of $\hat{x}_{T,h}$ is given by

$$V(\hat{x}_{T,h}) = E\left[(x_h - x_{T,h})(x_h - x_{T,h})'\right] + E\left[(\hat{x}_{T,h} - x_{T,h})(\hat{x}_{T,h} - x_{T,h})'\right]. \tag{18}$$

Considering the first term on the right hand side of (18), we have from (13) and (14) that

$$x_h - x_{T,h} = F_h^{-1} M_h \,\varepsilon_h \tag{19}$$

and, provided that ε_h satisfies the same covariance properties as ε, we obtain

$$E\left[(x_h - x_{T,h})(x_h - x_{T,h})'\right] = F_h^{-1} M_h \, I_{nh} \, M_h' \, F_h'^{-1} = F_h^{-1} \, \Omega_{nh} \, F_h'^{-1}, \tag{20}$$

where I_{nh} is the identity matrix of dimension $nh \times nh$ (which is the covariance matrix of ε_h), and $\Omega_{nh} = M_h M_h'$ is the block Toeplitz coveriance matrix.

Moving on to consider the expression $(\hat{x}_{T,h} - x_{T,h})$, it is convenient to eliminate the terms in ε and $\hat{\varepsilon}$ in Equations (14) and (15), respectively, by making use of (10) and (16). From Equation (10), we obtain

$$\varepsilon = M^{-1}Fx - M^{-1}Ez,$$

so that $x_{T,h}$ may be written

$$x_{T,h} = F_h^{-1}\left[(M^* M^{-1} F - F^*)\, x + (E^* - M^* M^{-1} E)\, z + E_h \, z_h\right]. \tag{21}$$

Similarly, from Equation (16), we obtain

$$\hat{\varepsilon} = \hat{M}^{-1}\hat{F}x - \hat{M}^{-1}\hat{E}z,$$

enabling us to write $\hat{x}_{T,h}$ as

$$\hat{x}_{T,h} = \hat{F}_h^{-1}\left[(\hat{M}^* \hat{M}^{-1} \hat{F} - \hat{F}^*)\, x + (\hat{E}^* - \hat{M}^* \hat{M}^{-1} \hat{E})\, z + \hat{E}_h \, z_h\right]. \tag{22}$$

A compact expression for the difference between (22) and (21) can be obtained by defining the $(nT + n^2 + n^s + mT + mh)$ – vector $w = [x'z'z_h']'$, which gives

$$\hat{x}_{T,h} - x_{T,h} = [\psi(\hat{\alpha}, \hat{y}) - \psi(\alpha, y)]\, w = [I_{nh} \otimes w']\, \mathrm{vec}\left[\psi(\hat{\alpha}, \hat{y})' - \psi(\alpha, y)'\right], \tag{23}$$

where $\psi(\alpha, y) = \left[F_h^{-1}(M^* M^{-1} F - F^*) : F_h^{-1}(E^* - M^* M^{-1} E) : F_h^{-1} E_h\right]$, a matrix of dimension $nh \times (nT + n^2 + n^s + mT + mh)$. Clearly, obtaining the asymptotic mean square error matrix of Equation (23) involves a function ψ of the vector of interest α. Since $T^{1/2}(\hat{\alpha} - \alpha)$ is asymptotically normal with mean vector 0 and covariance matrix Q, it follows that $T^{1/2}\left[\psi(\hat{\alpha}) - \psi(\alpha)\right]$

is aysmptotically normally distributed with mean vector zero and covariance matrix given by $\left[\frac{\partial\psi}{\partial\alpha'}\right] Q \left[\frac{\partial\psi}{\partial\alpha'}\right]'$. Applying these results to (23), we obtain

$$E\left[(\hat{x}_{T,h} - x_{T,h})(\hat{x}_{T,h} - x_{T,h})'\right] = T^{-1}\left[I_{nh} \otimes w'\right]\left[\frac{\partial \mathrm{vec}\psi}{\partial\alpha'}\right] Q \left[\frac{\partial \mathrm{vec}\psi}{\partial\alpha'}\right]' \left[I_{nh} \otimes w'\right]', \tag{24}$$

with the $(i,j)^{th}$ element of this matrix given by

$$T^{-1} w' \left[\frac{\partial\psi_i'}{\partial\alpha'}\right] Q \left[\frac{\partial\psi_j'}{\partial\alpha'}\right]' w,$$

where ψ_i denotes the i^{th} row of the matrix $\psi(\alpha)$. Combining Equations (20) and (24) gives the asymptotic mean square error matrix of the predictor $\hat{x}_{T,h}$:

$$V(\hat{x}_{T,h}) = F_h^{-1}\,\Omega_{nh}\,F_h'^{-1} + T^{-1}\left[I_{nh} \otimes w'\right]\left[\frac{\partial \mathrm{vec}\psi}{\partial\alpha'}\right] Q \left[\frac{\partial \mathrm{vec}\psi}{\partial\alpha'}\right]' \left[I_{nh} \otimes w'\right]'. \tag{25}$$

Note that this matrix gives the asymptotic mean square error for the predictions of the vectors $x_{T+1}, \ldots, x_{T+h}$, and not just x_{T+h} alone.

The steps involved in the computation of the AMSE matrix V may be summarised as follows:

(i) Compute $F_h^{-1}\Omega_{nh}F_h'^{-1}$ using the matrices Ω_0, Ω_1 and Ω_2 which will have been computed in the estimation of $\hat{\alpha}$ and $\hat{y}$. The matrix F_h^{-1} will already have been computed for constructing the forecasts $\hat{x}_{T,h}$.

(ii) Compute the matrix of derivatives $\frac{\partial \mathrm{vec}\psi}{\partial\alpha'}$ and the matrix $I_{nh} \otimes w'$.

(iii) Compute $V(\hat{x}_{T,h})$ using Equation (25) and making use of the matrices computed in steps (i) and (ii) and the matrix Q computed as part of the estimation algorithm.

The computation of the derivative matrix in step (ii) is best carried out numerically due to the complexity of the matrix function $\psi(\alpha)$.

Once the matrix $V(\hat{x}_{T,h})$ is computed, it will be of interest to compute confidence intervals for the forecasts. Let v_{ii} denote the i^{th} diagonal element of $V(\hat{x}_{T,h})$. Then the $(1-\alpha)\%$ asymptotic confidence interval for $x_{h,i}$ (the i^{th} element of x_h) is given by

$$Pr\{\hat{x}_{T,h,i} - z_{\alpha/2}\, v_{ii}^{1/2} \le x_{h,i} \le \hat{x}_{T,h,i} + z_{\alpha/2} v_{ii}^{1/2}\} = 1 - \alpha \tag{26}$$

where $z_{\alpha/2}$ is the critical value from the $N(0,1)$ distribution which puts $(\alpha/2)\%$ of the distribution in the upper tail. Bounds computed as in (26) would give policy makers some idea as to the expected range within which the true (future) value $x_{h,i}$ might lie.

5. CONCLUDING COMMENTS

This paper has further extended the results in [2] by deriving the asymptotic mean square error matrix of forecasts of discrete stock and flow data generated by a system of open second order stochastic differential equations. Two algorithms, one for computing the forecasts and one for computing the AMSE matrix, are presented which may be combined with the computationally efficient estimation algorithm given in [1]. The availability of a method for computing the AMSE matrix of the forecasts is essential for inferential purposes, in particular for the construction of confidence intervals for the predictions. Such algorithms are likely to prove useful now that continuous time methods are being employed more in empirical work in economics and finance. Indeed, the methods have recently been successfully applied by the author [6,7].

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